from $H_y(x, z)$ as follows:

$$E_x(x, z) = \frac{1}{\omega\epsilon_0\epsilon}\left(-i\epsilon_1\frac{\partial}{\partial z} - \epsilon_2\frac{\partial}{\partial x}\right)H_y(x, z) \quad (7a)$$

and

$$E_z(x, z) = \frac{1}{\omega\epsilon_0\epsilon}\left(i\epsilon_1\frac{\partial}{\partial x} - \epsilon_2\frac{\partial}{\partial z}\right)H_y(x, z). \quad (7b)$$

Let $H_y(x, z)$ and $E_x(x, z)$ be represented as a superposition of plane waves in the form

$$H_y(x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{H}_y(\zeta, z)e^{i\zeta x}d\zeta; \quad (8a)$$

$$E_x(x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{E}_x(\zeta, z)e^{i\zeta x}d\zeta. \quad (8b)$$

The use of (8) in (2), (7a) and (1) gives

$$\left[\frac{d^2}{dz^2} + \xi^2\right]\overline{H}_y(\zeta, z) = 0, \quad (9a)$$

$$\overline{E}_x(\zeta, z) = \frac{1}{\omega\epsilon_0\epsilon}\left(-i\epsilon_1\frac{\partial}{\partial z} - i\epsilon_2\zeta\right)\overline{H}_y(\zeta, z) \quad (9b)$$

and

$$\overline{E}_x(\zeta, 0) = E_0, \quad (9c)$$

where

$$\xi = \begin{cases} +\sqrt{k^2 - \zeta^2} & k > \zeta \\ +i\sqrt{\zeta^2 - k^2} & k < \zeta \end{cases}. \quad (10)$$

The solution of (9a) is obtained as

$$\overline{H}_y(\zeta, z) = Ae^{i\xi z} + Be^{-i\xi z}. \quad (11)$$

The application of the boundary conditions (9c) and $E_x(\zeta, a) = 0$ to (11) enables the determination of A and B with the following results:

$$A = -\frac{\omega\epsilon_0\epsilon E_0 e^{-i\xi a}}{2i\sin\xi a(\epsilon_1\xi - i\epsilon_2\zeta)};$$

$$B = -\frac{\omega\epsilon_0\epsilon E_0 e^{i\xi a}}{2i\sin\xi a(\epsilon_1\xi + i\epsilon_2\zeta)}. \quad (12)$$

It follows from (8a), (11) and (12) that

$$H_y(x, z) = \frac{i\omega\epsilon_0\epsilon E_0}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\sin\xi a}\cdot\left[\frac{e^{i\xi(z-a)}}{\epsilon_1\xi - i\epsilon_2\zeta} + \frac{e^{-i\xi(z-a)}}{\epsilon_1\xi + i\epsilon_2\zeta}\right]e^{i\zeta x}d\zeta. \quad (13)$$

The integrand of (13) is seen to have no branch points or poles at $\zeta = \pm k$. The poles of the integrand of (13) arise from the zeros of $\sin\xi a$ and those of $\epsilon_1\xi \pm i\epsilon_2\zeta$. The zeros of $\sin\xi a$ occur for $\zeta = \pm\sqrt{k^2 - (n\pi/a)^2}$, where n is an integer greater than zero. If $a < \pi/k$, these zeros are purely imaginary and the corresponding contribution to $H_y(x, z)$ will not give propagating modes. For $a < \pi/k$, the only singularities of the integrand in (13) are the poles given by the zeros of $\epsilon_1\xi \pm i\epsilon_2\zeta$. The zeros of $\epsilon_1\xi \pm i\epsilon_2\zeta$, which lie on the proper Riemann surface defined by (10), may be derived with the help of (4), to be given by

$$\zeta = \mp k_0\sqrt{\epsilon_1}\frac{|\epsilon_2|}{\epsilon_2}. \quad (14)$$

The contributions to $H_y(x, z)$ given by the residues of the poles (14) are obtained as

$$H_y(x, z) = \frac{\omega\epsilon_0|\epsilon_2|E_0}{2\sinh\left[k_0 a\frac{|\epsilon_2|}{\sqrt{\epsilon_1}}\right]}\cdot e^{ik_0x(|\epsilon_2|/\epsilon_2)\sqrt{\epsilon_1}+k_0(|\epsilon_2|/\sqrt{\epsilon_1})(a-z)} + \frac{\omega\epsilon_0|\epsilon_2|E_0}{2\sinh\left[k_0 a\frac{|\epsilon_2|}{\sqrt{\epsilon_1}}\right]}\cdot e^{-ik_0x}\left(\frac{|\epsilon_2|}{\epsilon_2}\right)\sqrt{\epsilon_1} + k_0\frac{\epsilon_2}{\sqrt{\epsilon_1}}(z - a). \quad (15)$$

It is evident that $H_y(x, z)$ given in (15) will give rise to a mode propagating in the x direction in the range of frequencies for which $\epsilon_1 > 0$. From (4) and Fig. 2, it is seen that $\epsilon_1 > 0$ for $0 < \Omega < R$ and $\Omega_2 < \Omega < \infty$. Also on substituting (15) in (7a) it is seen that $E_x(x, z) = 0$. Further, it is seen from (4) and Fig. 2, that $\epsilon_2 < 0$ for $0 < \Omega < R$ and that $\epsilon_2 > 0$ for $\Omega_2 < \Omega < \infty$. Therefore, the first and the second terms in (15) represent a TEM wave propagating, respectively, in the positive and the negative x directions in the frequency range $0 < \Omega < R$, and vice versa for $\Omega_2 < \Omega < \infty$.

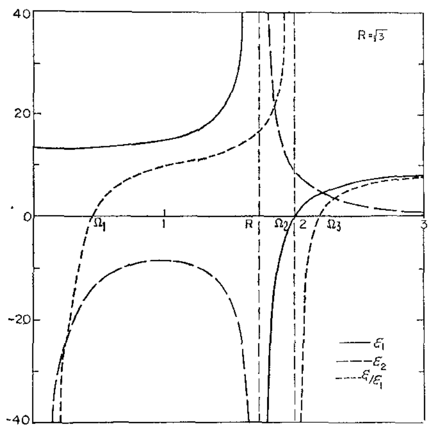

Fig. 2—ϵ_1, ϵ_2 and ϵ/ϵ_1 as a function of Ω.

In the frequency ranges for which $\epsilon_1 > 0$, it can be shown that $\sqrt{\epsilon_1} > \sqrt{\epsilon/\epsilon_1}$. It follows, therefore, from (3) that $k_0\sqrt{\epsilon_1} > k$. Since k is the wavenumber in an unbounded medium and $k_0\sqrt{\epsilon_1}$ is that of the TEM mode in the waveguide, it is clear that the TEM mode is a slow wave. In contrast to this, the TEM wave in the parallel-plate waveguide filled with an isotropic dielectric has the same phase velocity as in an unbounded medium.

It is well known that for the TEM mode in a parallel-plate waveguide filled with an isotropic dielectric, the field components are constant in amplitude in any cross section of the waveguide. But, for the TEM mode (15) obtained when the parallel-plate waveguide is filled with a gyrotropic dielectric, it is seen that for the first term in (15) the amplitude decreases exponentially from the bottom to the top plate whereas in the second term there is an exponential increase in amplitude.

As the distance a between the top and the bottom plates is increased, the infinity of higher order modes given by the poles $\zeta = \pm\sqrt{k^2 - (n\pi/a)^2}$ will be included one by one and, at the same time, the amplitude of the exponentially growing wave given by the second term in (15) falls off exponentially. Finally when the top plate is removed to infinity, the second term in (15) vanishes, the first term becomes the unidirectional surface wave along the bottom plate and the totality of the higher order modes combine to give the space wave [1].

In conclusion, it is appropriate to mention that a number of examples in the theory of propagation of electromagnetic waves in magnetoplasma slabs may be found in the literature such as in [2].

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S. R. SESHADRI
Gordon McKay Lab.
Harvard University
Cambridge, Mass.

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Boundary Excitation of Waveguides Containing Anisotropic Media*

Several methods can be utilized to launch electromagnetic waves along an ionized column contained in a cylindrical duct. An often used one is shown in Fig. 1 where the waveguide is aperture-coupled to a surrounding resonant cavity. In this configuration, the fields inside the waveguide are produced by boundary excitation, and the fields are uniquely determined by the assignment of the tangential component of $\boldsymbol{E}$ in the aperture. It is our purpose to present explicit equations for the various field components in terms of the value of $\boldsymbol{E}_{\text{tang}}$. The contribution from the volume sources $\boldsymbol{J}$ and $\boldsymbol{J}_m$ is also included for the sake of completeness. The detailed derivation follows the methods of Bresler and Marcuvitz[1,2] and is given elsewhere.[3]

To achieve a satisfactory degree of generality, we assume (Fig. 2) that the wave-

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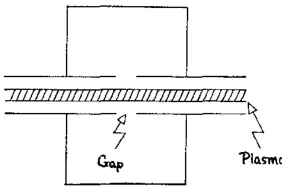

Fig. 1—A method of launching electromagnetic waves along an ionized column, using aperture coupling to resonant cavity.

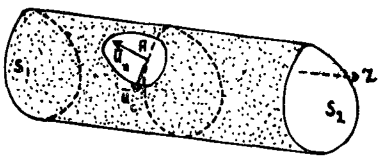

Fig. 2—Boundary-excited waveguide containing a plasma column.

guide is of arbitrary cross section, is excited through an aperture A of arbitrary shape and contains a "Hermitian" medium of tensorial characteristics

$$\epsilon(x, y) = \epsilon_t + \epsilon_3 \boldsymbol{u}_z \boldsymbol{u}_z$$
$$\mu(x, y) = \mu_t + \mu_3 \boldsymbol{u}_z \boldsymbol{u}_z. \quad (1)$$

Here, ϵ_t and μ_t are transverse Hermitian tensors. Generalization to non-Hermitian parameters is fairly easy, and involves use of adjoint transformations and bi-orthogonal sets of eigenvectors.

The z components of the fields can be eliminated from Maxwell's equations by use of the relationship

$$\text{curl}\, \boldsymbol{A} = (\text{grad}_t\, A_z x \boldsymbol{u}_z) - \frac{\partial}{\partial z}(\boldsymbol{A}_t x \boldsymbol{u}_z) + [\text{div}_t\, (\boldsymbol{A}_t x \boldsymbol{u}_z)]\boldsymbol{u}_z \quad (2)$$

where the subscript t indicates differentiation with respect to the (transverse) x, y coordinates. One obtains, for the transverse components,

$$\text{grad}_t \left[\frac{1}{j\omega\epsilon_3} \text{div}_t\, (j\boldsymbol{H}_t x \boldsymbol{u}_z)\right] + j\omega \boldsymbol{u}_z x(\mu_t \cdot j\boldsymbol{H}_t) - j\frac{\partial \boldsymbol{E}_t}{\partial z} = \text{grad}_t \left(\frac{J_z}{\omega\epsilon_3}\right) - \boldsymbol{u}_z x j \boldsymbol{J}_{mt} \quad (3)$$

$$\text{grad}_t \left[\frac{1}{j\omega\mu_3} \text{div}_t\, (\boldsymbol{E}_t x \boldsymbol{u}_z)\right] + j\omega \boldsymbol{u}_z x(\epsilon_t \cdot \boldsymbol{E}_t) - j\frac{\partial (j\boldsymbol{H}_t)}{\partial z} = \text{grad}_t \left(\frac{jJ_{mz}}{\omega\mu_3}\right) - \boldsymbol{u}_z x \boldsymbol{J}_t. \quad (4)$$

These equations can be put in the more concise form

$$\mathcal{L}(j\boldsymbol{H}_t) - j\frac{\partial \boldsymbol{E}_t}{\partial z} = \boldsymbol{\varrho}$$
$$\mathfrak{M}\boldsymbol{E}_t - j\frac{\partial}{\partial z}(j\boldsymbol{H}_t) = j\boldsymbol{\sigma} \quad (5)$$

where $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ are given vectors. In four-vector notation:

$$L\begin{pmatrix}\boldsymbol{E}_t\\ j\boldsymbol{H}_t\end{pmatrix} - j\frac{\partial}{\partial z}\begin{pmatrix}\boldsymbol{E}_t\\ j\boldsymbol{H}_t\end{pmatrix} = \begin{pmatrix}\boldsymbol{\varrho}\\ j\boldsymbol{\sigma}\end{pmatrix}. \quad (6)$$

The normal modes, in particular, are sourceless field distributions whose z dependence is of the form $e^{j\gamma z}$. We shall write their transverse components as $\boldsymbol{e}(x, y)e^{j\gamma z}$ and $\boldsymbol{h}(x, y)e^{j\gamma z}$. Evidently

$$L\begin{pmatrix}\boldsymbol{e}\\ j\boldsymbol{h}\end{pmatrix} + \gamma\begin{pmatrix}\boldsymbol{e}\\ j\boldsymbol{h}\end{pmatrix} = L(\alpha) + \gamma(\alpha) = 0 \quad (7)$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ satisfy the boundary conditions

$$\left.\begin{aligned} e_c &= 0\\ \text{div}\,(\boldsymbol{u}_z x \boldsymbol{h}) &= -j\omega\epsilon_3 e_z = 0 \end{aligned}\right\} \text{ on } (C). \quad (8)$$

Following Bresler and Marcuvitz, we utilize a Hermitian scalar product

$$\langle \alpha, \beta \rangle = \int [-\boldsymbol{a}^* \cdot (\boldsymbol{u}_z x \boldsymbol{d}) + \boldsymbol{b}^* : (\boldsymbol{u}_z x \boldsymbol{c})] dS \quad (9)$$

where

$$\alpha = \begin{pmatrix}\boldsymbol{a}\\ j\boldsymbol{b}\end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix}\boldsymbol{c}\\ j\boldsymbol{d}\end{pmatrix}.$$

Fundamental for our analysis is the relationship

$$\langle L\alpha, \beta \rangle = \langle \alpha, L\beta \rangle + \int_c \left[\frac{1}{\omega\epsilon_3} b_c^* \,\text{div}\,(\boldsymbol{u}_z x \boldsymbol{d}) - \frac{1}{\omega\epsilon_3} d_c\, \text{div}\,(\boldsymbol{u}_z x \boldsymbol{b}^*) + \frac{1}{\omega\mu_3} a_c^*\, \text{div}\,(\boldsymbol{u}_z x \boldsymbol{c}) - \frac{1}{\omega\mu_3} c_c\, \text{div}\,(\boldsymbol{u}_z x \boldsymbol{a}^*)\right] dC \quad (10)$$

in which the surface integrals are principal values if ϵ_3 or μ_3 happen to vanish at points inside the cross section. From (10) it follows immediately that the eigenvectors $\boldsymbol{\alpha}_m$ and $\boldsymbol{\alpha}_n$ with eigenvalues γ_m and γ_n satisfy the orthogonality condition

$$(\gamma_m^* - \gamma_n) \langle \boldsymbol{\alpha}_m, \boldsymbol{\alpha}_n \rangle = 0. \quad (11)$$

The four-vector

$$\begin{pmatrix}\boldsymbol{E}_t\\ j\boldsymbol{H}_t\end{pmatrix}$$

can be expanded in terms of the complete set of four-eigenvectors

$$\begin{pmatrix}\boldsymbol{e}_{\gamma m}\\ j\boldsymbol{h}_{\gamma m}\end{pmatrix}.$$

The expansion can be written in transmissionline form as

$$\boldsymbol{E}_t = \sum V_{\alpha m}(z)\boldsymbol{e}_{\alpha m} + \sum V_{\beta m}(z)\boldsymbol{e}_{\beta m} + \sum V_{\gamma m}(z)\boldsymbol{e}_{\gamma m} + \sum V_{\gamma m}{}^*(z)\boldsymbol{e}_{\gamma m}{}^*$$
$$\boldsymbol{H}_t = \sum I_{\alpha m}(z)(j\boldsymbol{h}_{\alpha m}) + \sum I_{\beta m}(z)(j\boldsymbol{h}_{\beta m}) + \sum I_{\gamma m}(z)(i\boldsymbol{h}_{\gamma m}) + \sum I_{\gamma m}{}^*(z)(j\boldsymbol{h}_{\gamma m}{}^*) \quad (12)$$

where α_m, β_m and γ_m denote respectively a real, an imaginary and a general complex eigenvalue. By utilizing (10) and (11) we arrive at the following basic equations for the expansion coefficients[4]

$$\frac{dV_{\alpha m}}{dz} - j\alpha_m I_{\alpha m} = -\frac{2}{N_{\alpha m}}\int_S J_z(\mathbf{e}^*_{\alpha m})_z dS - \frac{2}{N_{\alpha m}}\int_S \boldsymbol{J}_m \cdot \boldsymbol{h}^*_{\alpha m} dS + \frac{2}{N_{\alpha m}}\int_c E_z(\boldsymbol{h}^*_{\alpha m})_c dc \quad (13)$$

$$\frac{dI_{\alpha m}}{dz} - j\alpha_m V_{\alpha m} = -\frac{2}{N_{\alpha m}}\int_S \boldsymbol{J} \cdot \mathbf{e}^*_{\alpha m} dS - \frac{2}{N_{\alpha m}}\int_S J_{mz}(\boldsymbol{h}^*_{\alpha m})_z dS - \frac{2}{N_{\alpha m}}\int_c E_c(\boldsymbol{h}^*_{\alpha m})_z dc \quad (14)$$

$$\frac{dV_{\beta m}}{dz} + \beta_m I_{\beta m} = +\frac{2}{N_{\beta m}}\int_S J_z(\mathbf{e}^*_{\beta m})_z dS + \frac{2}{N_{\beta m}}\int_S \boldsymbol{J}_m \cdot \boldsymbol{h}^*_{\beta m} dS - \frac{2}{N_{\beta m}}\int_c E_z(\boldsymbol{h}^*_{\beta m})_c dc \quad (15)$$

$$\frac{dI_{\beta m}}{dz} + \beta_m V_{\beta m} = -\frac{2}{N_{\beta m}}\int_S \boldsymbol{J} \cdot \mathbf{e}^*_{\beta m} dS - \frac{2}{N_{\beta m}}\int_S J_{mz}(\boldsymbol{h}^*_{\beta m})_z dS - \frac{2}{N_{\beta m}}\int_c E_c(\boldsymbol{h}^*_{\beta m})_z dc \quad (16)$$

$$\frac{dV_{\gamma m}}{dz} - j\gamma_m I_{\gamma m} = -\frac{2}{N_{\gamma m}}\int_S J_z(\mathbf{e}^*_{\gamma m^*})_z dS - \frac{2}{N_{\gamma m}}\int_S \boldsymbol{J}_m \cdot \boldsymbol{h}^*_{\gamma m^*} dS + \frac{2}{N_{\gamma m}}\int_c E_z(\boldsymbol{h}^*_{\gamma m^*})_c dc \quad (17)$$

$$\frac{dI_{\gamma m}}{dz} - j\gamma_m V_{\gamma m} = -\frac{2}{N_{\gamma m}}\int_S \boldsymbol{J} \cdot \mathbf{e}^*_{\gamma m^*} dS - \frac{2}{N_{\gamma m}}\int_S J_{mz}(\boldsymbol{h}^*_{\gamma m^*})_z dS - \frac{2}{N_{\gamma m}}\int_c E_c(\boldsymbol{h}^*_{\gamma m^*})_z dc \quad (18)$$

$$\frac{dV_{\gamma m^*}}{dz} - j\gamma^*_m I_{\gamma m^*} = -\frac{2}{N^*_{\gamma m}}\int_S J_z(\mathbf{e}^*_{\gamma m})_z dS - \frac{2}{N^*_{\gamma m}}\int_S \boldsymbol{J}_m \cdot \boldsymbol{h}^*_{\gamma m} dS + \frac{2}{N^*_{\gamma m}}\int_c E_z(\boldsymbol{h}^*_{\gamma m})_c dc \quad (19)$$

$$\frac{dI_{\gamma m^*}}{dz} - j\gamma^*_m V_{\gamma m^*} = -\frac{2}{N^*_{\gamma m}}\int_S \boldsymbol{J} \cdot \mathbf{e}^*_{\gamma m} dS - \frac{2}{N^*_{\gamma m}}\int_S J_{mz}(\boldsymbol{h}^*_{\gamma m})_z dS - \frac{2}{N^*_{\gamma m}}\int_c E_c(\boldsymbol{h}^*_{\gamma m})_z dc. \quad (20)$$

A normalization factor such as $N\gamma_m$ stands for

$$\left\langle\begin{matrix}\boldsymbol{e}_{\gamma m^*}\\ j\boldsymbol{h}_{\gamma m^*}\end{matrix}\quad\begin{matrix}\boldsymbol{e}_{\gamma m}\\ j\boldsymbol{h}_{\gamma m}\end{matrix}\right\rangle.$$

Eqs. (13) to (20) represent our main result. The second members show clearly the separate contributions from the boundary and volume sources. The form of the equations is seen to be particularly simple; it is only in the computation of the eigenvalues and eigenvectors that the anisotropies and inhomogeneities of the medium give rise to difficulties. We notice that a complete solution of the differential equations requires consideration of the boundary conditions at the terminal surfaces S_1 and (or) S_2.

J. VAN BLADEL†
Division of Theor. Electr. Engrg.
Royal Inst. of Technology,
Stockholm, Sweden

[4] $(\boldsymbol{e}^*_{\gamma m^*})_z$ stands for the z component of the complex conjugate of $\boldsymbol{e}_{\gamma m^*}$, eigenvector relative to γ_m^* (which is an eigenvalue if γ_m is an eigenvalue).

† On leave from the Department of Electrical Engineering, University of Wisconsin, Madison.